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MINIMIZATION OF THE INTEGRAL ESTIMATE OF THE KINETIC ENERGY OF A HARMONIC OSCILLATOR BY AN IMPULSE CONTROL

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The kinetic energy of the transient response of a harmonic oscillator is minimized by actions with a bounded current total momentum. Such a problem arises, for example, when choosing a mass flow program minimizing the kinetic energy of transfer of a satellite into a circular orbit by a reactive force having the direction of the Earth's gravitational force (see [1], p. 32). It is shown that the optimal control contains an impulse component. This leads to the violation of the optimality principle for extremals. Therefore, the synthesis procedure is based on the analysis of an auxiliary variational problem under the usual constraints on the control [2].

1. Statement of the problem and its reduction. Let the plant be described by the differential equation of a controlled harmonic oscillator

$$x^{\cdot\cdot} + \omega^2 x = ku \quad (\omega, k \neq 0) \tag{1.1}$$

(where u is the control)

$$u(t) = 0\,(t < 0);\ |\nu[u](t)| \leqslant 1,\ \nu[u](t) = \int_{-\infty}^{t} u(\tau)\,d\tau \tag{1.2}$$

Here $\nu[u](t)$ is a quantity proportional to the current value of the total momentum of the control force. We examine the autput $x[x_0, x_0^{\cdot}; u](t)$ of plant (1.1), corresponding to the initial conditions

$$x[x_0, x_0^{\cdot}; u](0) = x_0, \qquad x^{\cdot}[x_0, x_0^{\cdot}; u](0) = x_0^{\cdot} \tag{1.3}$$

and to some program $u(t)$, subject to requirements (1.2). We define the control's performance index

$$\Delta[x_0, x_0, u] = \int_0^{\infty} \{x^{\cdot}[x_0, x_0^{\cdot}, u](t)\}^2\,dt \tag{1.4}$$

The following problem is solved in this paper.

Problem A. In class (1.2) find the control minimizing functional (1.4) computed on the motion of plant (1.1) with initial conditions (1.3).

A strict mathemetical statement of Problem A has been given in [2]. Since the admissible controls are generalized functions [2], the classical variational principles written out for Problem A lose sense. This circumstance reflects a deeper fact; as it will be stated, the extremals of Problem A do not satisfy the principle of optimality [3]. In this paper, as in [2], such an obstacle is overcome by passing to an auxiliary variational problem with the usual constraints on the control.

Before we formulate this problem we note that without loss of generality we can set $\omega = k = 1$. Then in the standard notation $x_1 = x,\ x_2 = x^{\cdot}$ plant (1.1) can be described by the system

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_1 + u; \quad x_1(0) = x_{10} = x_0, \quad x_2(0) = x_{20} = x_0^{\cdot} \tag{1.5}$$

and functional (1.4) receives the form

$$\Delta\,[x_{10}, x_{20}; u] = \int_0^\infty x_2^2(t)\,dt\ . \tag{1.6}$$

We define a transformation of the phase trajectories (but not of the coordinates) of system (1.5) by setting

$$x_1 = \mu_2, \ x_2 = \nu - \mu_1 \tag{1.7}$$

In [2] it was proved that such transformed trajectories are described by the system

$$\mu_1^{\cdot} = \mu_2, \quad \mu_2^{\cdot} = \nu - \mu_1; \quad \mu_1(0) = \mu_{10} = -x_{20}, \quad \mu_2(0) = \mu_{20} = x_{10} \tag{1.8}$$

while to the functional (1.6) there corresponds the index

$$\Delta\,[\mu_{10}, \mu_{20}; \nu] = \int_0^\infty (\nu - \mu_1)^2\,dt \tag{1.9}$$

Thus, Problem A is equivalent to the following auxiliary problem.

Problem B. In the class (1.2) of control signals ν'(the first of conditions (1.2) signifies that $\nu(t) = 0$ when $t < 0$) find the program realizing the minimum of functional (1.9), computed on the trajectories of system (1.8).

The theorem stated below follows from a well-known generalization of the Weierstrass theorem.

Theorem 1. The solution of Problem B exists and is unique.

2. Synthesis of the optimal control for the auxiliary problem.

We apply the maximum principle [4] to solve Problem B. According to this principle the optimal function $\nu(t)$ satisfies the relation

$$\max_{|\nu|\leqslant 1} H\,[\mu_1(t), \mu_2(t);\ \psi_1(t), \psi_2(t); \nu] = H\,[\mu_1(t), \mu_2(t);\ \psi_1(t), \psi_2(t);\ \nu(t)] = 0 \tag{2.1}$$

$$H\,[.,.;.,.;.] = -(\nu - \mu_1)^2 + \psi_1\mu_2 + \psi_2(\nu - \mu_1) \tag{2.2}$$

Here ψ_1, ψ_2 is a nontrivial solution of the adjoint system

$$\psi_1^{\cdot} = \psi_2 + 2\,[\mu_1 - \nu(t)], \quad \psi_2^{\cdot} = -\psi_1; \quad \psi_1(0) = \psi_{10}, \quad \psi_2(0) = \psi_{20} \tag{2.3}$$

Let $t_\alpha \geqslant 0$ and let $|\nu(t)| = 1$ in some neighborhood of t_α. Then from (2.1),

$\nu(t) = \operatorname{sign} \nu^\circ(t)$, $\nu^\circ = \mu_1 + (\psi_2 / 2)$, ψ_1, ψ_2 is a solution of system (2.3) with initial conditions $\psi_1(t_\alpha)$, $\psi_2(t_\alpha)$. Under the action of this control the phase point of system (1.8) moves along the circle

$$[\mu_1 - \nu(t_\alpha)]^2 + \mu_2^2 = [\mu_1(t_\alpha) - \nu(t_\alpha)]^2 + \mu_2^2(t_\alpha) \tag{2.4}$$

moreover, identically

$$-[\nu(t_\alpha) - \mu_1]^2 + \psi_1\mu_2 + \psi_2[\nu(t_\alpha) - \mu_1] = 0 \tag{2.5}$$

Let t_β be a root of the equation

$$\nu^\circ(t) = \nu(t_\alpha) \tag{2.6}$$

other than t_α. According to (2.1), a switching occurs at the instant t_β on the control $\nu(t) = \nu^0(t)$, where ψ_1, ψ_2 is a solution of the system

$$\psi_1^\cdot = 0, \qquad \psi_2^\cdot = -\psi_1 \tag{2.7}$$

with initial conditions $\psi_1(t_\beta)$, $\psi_2(t_\beta)$. For $t > t_\beta$, identically

$$\psi_1(t_\beta)\mu_2 + (\psi_2/2)^2 = 0 \tag{2.8}$$

since $\psi_1 = \text{const}$ by virtue of (2.7). Determining ψ_2 from (2.8) and entering it into the formula for ν°, we obtain

$$\nu(t) = \mu_1 + \sqrt{-\psi_1(t_\beta)\mu_2}\ \operatorname{sign}\psi_2 \tag{2.9}$$

With due regard to (2.9) the plant's motion itself is described by the system

$$\mu_1^\cdot = \mu_2, \quad \mu_2^\cdot = \sqrt{-\psi_1(t_\beta)\mu_2}\ \operatorname{sign}\psi_2 \tag{2.10}$$

By the definition of instant t_β and according to (2.9), the phase point of system (1.8) is located at the switching instant on the line

$$\mu_1 + \sqrt{-\psi_1(t_\beta)\mu_2}\ \operatorname{sign}\psi_2(t_\beta) = \nu(t_\beta) \tag{2.11}$$

Its subsequent motion is effected, according to (2.10), along the semi-cubical parabolas'

$$\mu_1 - \mu_1(t_\beta) = \frac{2}{3}\frac{\operatorname{sign}\psi_2(t)}{\sqrt{-\psi_1(t_\beta)}}[\mu_2^{3/2} - \mu_2^{3/2}(t_\beta)], \quad \mu_2(t_\beta) > 0 \tag{2.12}$$

$$\mu_1 - \mu_1(t_\beta) = \frac{2}{3}\frac{\operatorname{sign}\psi_2(t)}{\sqrt{\psi_1(t_\beta)}}[(-\mu_2)^{3/2} - (\mu_2(t_\beta))^{3/2}], \quad \mu_2(t_\beta) < 0 \tag{2.13}$$

Two cases are further possible. In the first of them we have $\psi_2(t_\gamma) = 0$ at some instant t_γ. Then from (2.8), $\mu_2(t_\gamma) = 0$ and, since $|\nu(t_\gamma)| \leqslant 1$, by virtue of (2.9) $|\mu_1(t_\gamma)| \leqslant 1$. For $t > t_\gamma$, from (2.1) we have $\nu(t) = \mu_1(t_\gamma)$, and under the action of this control the phase point is located at the equilibrium position. In the second case ψ_2 does not change sign. The derivative of (2.9) by virtue of (2.10) has the form $\nu^\cdot|_{(2.10)} = \mu_2 - {}^1/_2\,\psi_1(t_\beta)$. According to (2.8), μ_2 and $\psi_1(t_\beta)$ are opposite in sign. Therefore, the derivative being discussed is sign-constant. Further, the derivative of ν° by virtue of (1.8) and (2.3) equals $\nu^\cdot|_{(2.10)}(t_\beta)$ at instant t_β and has the sign of the number $-\nu(t_\beta)$. Consequently $\nu^\cdot|_{(2.10)}(t)\ \nu(t_\beta) < 0$. This inequality allows us to conclude that at some instant t_γ the phase point of system (1.8) is found on the line

$$\mu_1 + \sqrt{-\psi_1(t_\beta)\mu_2}\ \operatorname{sign}\psi_2(t_\beta) = -\nu(t_\beta) \tag{2.14}$$

Thus, the extremal of Problem B consists of arcs of the circles (2.4) with alternating centers (−1, 0), (1, 0). These arcs are separated by parts of parabolas (2.12), (2.13). By the same token, Problem B has been reduced to the ascertainment of the location of the original position on the extremal and to the determination of the initial conditions of the adjoint system, corresponding to this extremal.

The first of these questions is answered for the initial conditions

$$1 \leqslant \mu_{10}, \qquad \mu_{20} = 0 \tag{2.15}$$

Assuming $|\nu(0)| < 1$, from (2.8) we can obtain $\psi_{20} = 0$. This, with due regard to (2.9), leads to the bound $|\mu_{10}| < 1$, contradicting (2.15). Consequently, the relation (2.5) should be fulfilled at the initial instant; whence it follows that $\nu(0) = 1$ and

$$\psi_{20} = 1 - \mu_{10} \tag{2.16}$$

The projection onto the (μ_1, μ_2)-plane of the trajectory of systems (1.8), (2.3) with initial conditions satisfying relations (2.15), (2.16), we shall call the ψ_{10}-trajectory. The phase point of system (1.8) moves along ψ_{10}-trajectory, at first turning around the point (1, 0). The switching instant t_1 on the nonlinear control (2.9) $(t_\beta = t_1)$ is determined by Eq. (2.6) $(t_\alpha = 0,\ \nu(t_\alpha) = 1)$. Subsequent motion is effected along parabola (2.13) until hitting either on the segment $|\mu_1| \leqslant 1,\ \mu_2 = 0$, or on line (2.14) $(t_\beta = t_1,\ \nu(t_\beta) = 1)$ of the second program switching. From the instant $t_2 = t_\gamma$ the phase point starts to rotate around the point (−1, 0). On this segment, the solution of system (1.8), (2.3), calculated by the Cauchy formula, has the form

$$\mu_1 + 1 = \mu_{22}(l_2 \cos\tau + \sin\tau), \quad \mu_2 = \mu_{22}(\cos\tau - l_2 \sin\tau) \tag{2.17}$$

$$\psi_1 = (\tau - l_2)(\mu_1 + 1), \qquad \psi_2 = -2(\mu_1 + 1) + \tau\mu_2 + \mu_{22}(1 + l_2^2)\sin\tau$$

Here

$$\tau = t - t_2, \quad \mu_{i2} = \mu_i(t_2) \quad (i = 1, 2), \quad \mu_{12} + 1 = l_2\mu_{22}$$

The switching instant $t_3 = t_2 + \tau_3$ on the nonlinear control is determined by Eq. (2.6) $(t_\alpha = t_2, \nu(t_\alpha) = 1)$. The substitution of the first and last of solutions (2.17) into this equation yields the relation

$$\operatorname{ctg}\tau = l_2 - (1 + l_2^2)\tau^{-1} \tag{2.18}$$

Let $\varphi_2 = \operatorname{arctg} l_2$. Since the zero of the right-hand side of (2.18) $\operatorname{tg}\varphi_2 + \operatorname{ctg}\varphi_2 > 2 > \pi/2$, the following statement is valid.

Lemma 1. If $0 < \varphi_2 < \pi/2$, then $\pi/2 < \tau_3 < \pi$. The estimate $(\pi/2) - \varphi_2 < \tau_3 < \pi$ holds for $-\pi/2 < \varphi_2 < 0$.

Corollary. Along a ψ_{10}-trajectory there are no more than two switchings in each half-space $\mu_2 < 0,\ \mu_2 > 0$.

Let $\psi_{10}(\mu_{10} - 1)$ be the value of ψ_{10} corresponding to the optimal ψ_{10}-trajectory. As μ_{10} sweeps from 1 to ∞ the points $(\mu_{1i}, \mu_{2i}) = (\mu_1(t_i), \mu_2(t_i))$ $(i = 1, 2)$, the $\psi_{10}(\mu_{10} - 1)$-trajectories describe the lines

$$\mu_1 + s_i(\mu_2) = (-1)^{i+1} \quad (i = 1, 2) \tag{2.19}$$

respectively. According to the Corollary to Lemma 1 there are no other switching lines of the optimal control synthesis problem for system (1.8) in the region $\mu_2 < 0$. By symmetry considerations the switching lines in region $\mu_2 > 0$ have the form

$$\mu_1 - s_i(-\mu_2) = -(-1)^i \quad (i = 1, 2) \tag{2.20}$$

The next assertion is established by means of Lemma 1 and its Corollary.

Lemma 2. $s_i(\mu_2) > 0$ for $\mu_2 < 0$ $(i = 1, 2)$

Lemma 3. The optimal transient response in system (1.8) is unbounded in time.

In fact, assuming that the transient response time t_γ is finite, according to [4] we have $\psi_1(t_\gamma) = \psi_2(t_\gamma) = 0$. Since the response can be completed only in the nonlinear control mode, by virtue of (2.7) $\psi_1(t) = 0$ for $t_1 \leqslant t \leqslant t_\gamma$. Consequently, on the interval $[t_1, t_\gamma]$ the extremal is a straight line parallel to the axis $\mu_2 = 0$ and cannot intersect it.

Now, with the aid of Lemmas 3 and 2 we can derive the equation in superpositions relative to s_1 as the equation of invariant curves. Further, alowing for (2.19) $(i = 1)$ in (2.11), we obtain

$$\psi_{11} = \psi_1(t_\beta) = -\mu_{21}^{-1} s_1^2(\mu_{21}) \tag{2.21}$$

According to (2.7), ψ_1 does not vary on a segment of nonlinear control (2.9). Consequently, this function is a realization of the function $\Psi_-(\mu_1, \mu_2)$ defined by the equation

$$\sqrt{-\mu_2}\,\partial\Psi_-/\partial\mu_1 - \sqrt{\Psi_-}\,\partial\Psi_-/\partial\mu_2 = 0 \tag{2.22}$$

with initial conditions (2.21) on line (2.19) $(i = 1)$. Equation (2.22) has the following integrals:

$$c_1 = \Psi_-, \qquad c_2 = 1 - \mu_1 + {}^2/_3\,\Psi_-^{-1/2}(-\mu_2)^{3/2} \tag{2.23}$$

Substituting (2.21) and (2.19) $(i = 1)$ into them, with due regard to Lemma 2, we obtain a third-degree equation in μ_2, solving which by Cardan's formulas, we have

$$\mu_2^\circ = {}^1/_2\,(c_{1/2})^{1/3}\,[2^{1/3}c_1^{2/3} - (3c_2 + \sqrt{2c_1^2 + 9c_2^2})^{2/3} - (3c_2 - \sqrt{2c_1^2 + 9c_2^2})^{2/3}]$$

The second relation, necessary for obtaining the connection between c_1 and c_2, has the form $-\mu_2^{\circ-1} s_1^2(\mu_2^\circ) = c_1$. A subsequent substitution of integrals (2.23) into these two formulas yields an equation determining the desired function Ψ_-. In region $\mu_2 > 0$ the function being discussed has the form $-\Psi_-(-\mu_1, -\mu_2)$. Now it is not difficult to complete the synthesis of the optimal control. The extremals of Problem B are shown in Fig. 1. Here, *1* and *2* are the lines (2.19) $(i = 1, 2)$; *3* and *4* are the lines (2.20) $(i = 1, 2)$.

3. Questions of approximation.

Below we shall need an explicit form for the solution of systems (1.8), (2.3) with initial conditions (2.15), (2.16)

$$\mu_1 - 1 = (\mu_{10} - 1)\cos t, \tag{3.1}$$
$$\mu_2 = -(\mu_{10} - 1)\sin t$$
$$\psi_1 = t(\mu_1 - 1) + \psi_{10}\cos t,$$
$$\psi_2 = -\mu_1 + 1 + t\mu_2 - \psi_{10}\sin t$$

The substitution of the first and last of solutions (3.1) into (2.6) yields the equation

$$\operatorname{ctg} t = l_0 + t, \quad l_0 = (\mu_{10} - 1)^{-1}\psi_{10} \tag{3.2}$$

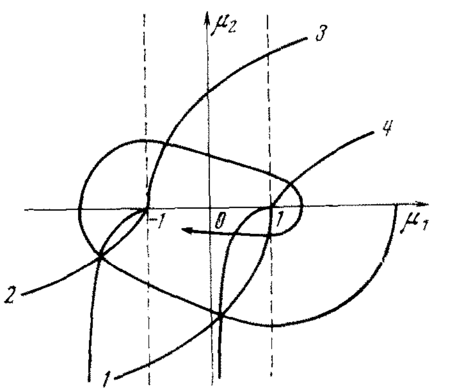

Fig. 1

The following assertion is proved in the

Appendix (Sect. 5).

Lemma 4. Let $\psi_{10}^{(1)} > \psi_{10}^{(2)}$. Then the $\psi_{10}^{(1)}$-trajectory, bordering the $\psi_{10}^{(2)}$-trajectory, recedes from it.

Let us separate the real axis into three sets. To set M_1 we refer those ψ_{10} for which the ψ_{10}-trajectories have a finite transient response time. Values of ψ_{10} defining ψ_{10}-trajectories not winding to the segment $|\mu_{10}| \leqslant 1$, $\mu_{20} = 0$, comprise set M_3. All the remaining go into M_2. The next statement follows from Lemmas 3 and 4 and Theorem 1.

Theorem 2. The sets M_i $(i = 1, 3)$ are continuous; M_2 consists of the one point $\psi_{10}(\mu_{10} - 1)$. The corresponding $\psi_{10}(\mu_{10} - 1)$-trajectory is an extremal of Problem B with initial conditions (2.15).

An analysis of Eq. (3.2) with due regard to Lemma 2 yields the estimate

$$\psi_{10}(\mu_{10} - 1) < \pi/2\,(\mu_{10} - 1)$$

In Theorem 2 we have established an algorithm for approximating $\psi_{10}(\mu_{10} - 1)$. We describe it. Let $1 \leqslant \mu_{10} \leqslant 3$. Then the ψ_{10}-trajectory with

$$\begin{aligned} \psi_{10}^{(1)}(\mu_{10} - 1) &= (\mu_{10} - 1)(\operatorname{ctg} t_1 - t_1) \\ t_1 &= \arccos\,[\sqrt{9(\mu_{10} - 1)^2 - 2} - 3(\mu_{10} - 1)^{-1}] \end{aligned} \tag{3.3}$$

in a half revolution falls into the position (−1, 0). The value of (3.3) is obtained from (3.2) where t is the number found from the condition that the point (−1, 0) lies on parabola (2.13) with

$$\operatorname{sign}\psi_2 = 1, \qquad \sqrt{\psi_1(t_\beta)} = (1 - \mu_{11})(-\mu_{21})^{-1/2}$$

while μ_{11}, μ_{21} are defined according to (3.1). It can be shown that it is impossible to fall from the position with $\mu_{10} > 3$ into the point (−1, 0) in a half revolution and that for $1 < \mu_{10} < 5$ there exists $\psi_{10}^{(2)}(\mu_{10} - 1)$ for which the ψ_{10}-trajectory falls into the point (1, 0) in one revolution, etc. Thus, let $2k - 1 < \mu_{10} < 2k + 1$ and $\psi_{10}^{(n)}(\mu_{10} - 1)$ $(n = k, k + 1, \ldots)$ be such that the corresponding ψ_{10}-trajectory falls into the position $((-1)^n, 0)$ after n half-revolutions. From Theorem 2 follows a corollary.

Corollary. The sequence $\psi_{10}^{(n)}(\mu_{10} - 1)$ by increasing, tends to $\psi_{10}(\mu_{10} - 1)$ as $k \leqslant n \to \infty$.

We now describe an algorithm for approximating the switching lines (2.19). At first we construct the line on which the phase point falls into one of the ends of the segment $|\mu_1| \leqslant 1$, $\mu_2 = 0$ under the action of control (2.9) wherein $\psi_1(t_\beta)$ has been found at the initial instant from condition (2.11). We have the ellipse

$$\mu_1^2 + {}^2/_3\,\mu_2^2 = 1 \tag{3.4}$$

Further, in the region $\mu_1 < -1$, $\mu_2 < 0$ we obtain the line on which the phase point under the action of control $\nu = -1$ hits onto ellipse (3.4) as onto a switching line. Let $\tau^{(1)}$ be the time of motion of the point from the desired line to the axis $\mu_2 = 0$ and let $\tau^{(2)}$ be the remaining time of motion to ellipse (3.4). With the aid of the first two of solutions (2.17) we can obtain the values

$$\begin{aligned} &\tau^{(1)} = \operatorname{arc\,tg}(\mu_1 + 1)^{-1}\mu_2, \tau^{(2)} = \arccos\,(\sqrt{9 - 2R^2} - 3)\,R^{-1} \\ &R^2 = (\mu_1 + 1)^2 + \mu^2{}_2 \end{aligned}$$

The expression for $\tau^{(2)}$ has sense for $R \leqslant 2$. Substituting $\tau = \tau^{(1)} + \tau^{(2)}$ into (2.18) we have the equation for the desired line

$$\operatorname{ctg} \tau^{(1)} + \operatorname{ctg} \tau^{(2)} = \tau^{(1)} + \tau^{(2)} \tag{3.5}$$

This line, starting at point (−1, 0), makes a right angle with the axis $\mu_2 = 0$ at the point (−3, 0). Let us now find, in the region $\mu_1 < 1$, $\mu_2 < 0$, the line of the preceding switching on control (2.9). For this we solve a third-degree equation in $\mu_1 (t_\beta)$, obtained from (2.13) with due regard to the fact that (2.14) is fulfilled for the points of line (3.5), while (2.11) is fulfilled for the points of the desired line. We have

$$\begin{aligned} &\mu_1 (t_\beta) = 1 + \sqrt{-q + \sqrt{q^2 + p^3}} + \sqrt{-q - \sqrt{q^2 + p^3}} \\ &\mu_2 (t_\beta) = (\mu_1 + 1)^{-2} \mu^2 [\mu_1 (t_\beta) - 1]^2 \\ &p = -{}^2/_9 (\mu_1 + 1)^2 \mu_2^{-1} \\ &q = -{}^1/_2 (\mu_1 + 1)^3 + {}^1/_3 (\mu_1 + 1)^2 (\mu_1 - 1) \mu_2^{-1} \end{aligned} \tag{3.6}$$

The μ_1, μ_2 occurring in (3.6) are connected by Eq. (3.5) and, therefore, (3.6) are the parameteric equations of the desired line. The latter, starting at the point (1, 0), is tangent to line (3.5) at the point (−3, 0). Lines (3.5) and (3.6), as well as those centrally symmetric to them, form the first approximation to lines (2.19).

Having chosen, instead of ellipse (3.4), the first-approximation line lying in the region $\mu_2 > 0$ and starting at point (−1, 0), and having effected the subsequent constructions, we obtain the second approximation to lines (2.19), etc. (see Fig. 2). It can be

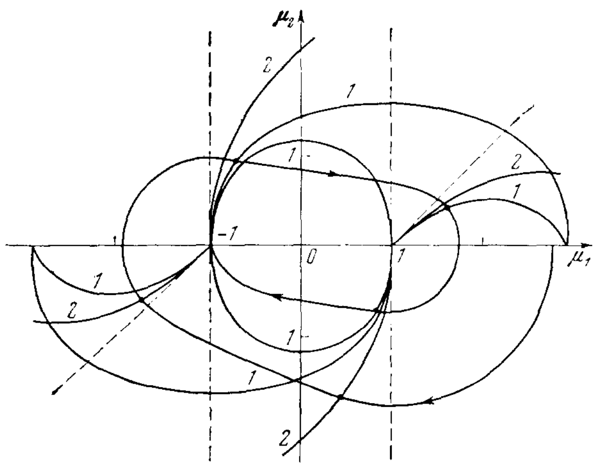

Fig. 2

established that the n-th approximation line:

1) leaves from the ends of the segment $|\mu_1| \leqslant 1$, $\mu_2 = 0$;

2) makes one and the same angle with the axis $\mu_2 = 0$ at the point $(-2n - 1, 0)$ in region $\mu_2 < 0$ and at the point $(2n + 1, 0)$ in region $\mu_2 > 0$;

3) lies below (above) the $(n - 1)$-st approximation lines $(n \geqslant 2)$ in the half-plane $\mu_2 < 0$ $(\mu_2 > 0)$.

By property (3) and the Corollary to Theorem 2 the sequence obtained converges to the optimal switching lines in Problem B.

Let $\omega_-(l_2)$ be the largest negative root of Eq. (3.2).

Lemma 5. The line (2.19) $(i = 1)$ makes a right angle at point (1, 0), while the slope of the line (2.19) $(i = 2)$ at point $(-1, 0)$ equals $-\operatorname{tg}\omega_-(-\pi/2)$.

In fact, the first-approximation line is tangent to the ray $\mu_1 = 1$, $\mu_2 \leqslant 0$ at the point (1, 0), while by property (3) and Lemma 2 the line (2.19) $(i = 1)$ lies between them. Further, an investigation of Eq. (3.2) with due regard to Lemma 2 shows that $\omega_-(-\pi/2)$. This signifies that the ray with origin at point (1, 0) and slope $-\operatorname{tg}\omega_-(l_0) >$ 2) lies above the line (2.20) $(i = 2)$. On the basis of property (3) the latter is located above the corresponding first-approximation line which has the slope $-\operatorname{tg}\omega_-(-\pi/_2)$ at point (1, 0).

4. Synthesis of the optimal control for the original problem.

Transforming Eqs. (2.19) by formulas (1.7) we obtain the switching lines for the original problem

$$s_i(x_1) = x_2 \qquad (i = 1, 2) \tag{4.1}$$

By Lemma 2 lines (4.1) are located in the second quadrant, and according to Lemma 5, the first of them is tangent to the x_1-axis, while the second one is tangent to the straight line $x_2 = \operatorname{ctg}\omega_-(-\pi/2)\, x_1$, above which it lies. Two other switching lines are centrally symmetric to lines (4.1).

We describe the extremal with the initial condition $x_{10} < 0$ The case $x_{10} > 0$ is analogous to the one selected. At first let $s_1(x_{10}) > x_{20} + 1$. Then the point (μ_{10}, μ_{20}), corresponding by virtue of rule (1.8) for recalculating the initial conditions, is located in region $\mu_2 < 0$ to the right of the line (2.19) $(i = 1)$. According to Sect. 2, $\nu = 1$. Since $\nu = 0$ for $t < 0$, the control $u = \nu^{\cdot}$ contains an impulse component which causes an instantaneous displacement of the phase point of system (1.5) into the position $(x_{10}, x_{20} + 1)$. After this it reaches, by rotating around the origin, the line (4.1) $(i = 1)$ at instant t_β. For $t > t_\beta$, ν is determined by formula (2.9) whose differentiation by virtue of system (2.10) yields the equation $u = \mu_2 - {}^1/_{2\,\beta})$. Allowing for (2.21) and (1.7) in this expression, we obtain $u = x_1 + {}^1/_2\, x_{11}^{-1} s_1^2(x_{11})$, where x_{11} is the abscissa of the first switching point. This control causes a motion along the parabola $x_1 = x_{11} s_1^{-2}(x_{11})\, x_2^2$ up to the instant t_γ. At this instant the phase point of system (1.8) reaches the line (2.19) $(i = 2)$, while, respectively, the extremal of system (1.5) reaches the line (4.1). Next, $\nu = -1$, is established, as a result of which the phase point of system (1.5), having begun a rotation around the origin, leaves the region being considered.

If $x_{20} > s_2(x_{10}) + 1$, we can discern that after the initial displacement by unity downward the phase point of system (1.5) departs into the region $x_{10} > 0$ along a circle. It remains to analyze the case when $s_1(x_{10}) - 1 < x_{20} < s_2(x_{10}) + 1$. By applying the recalculation formulas (1.8) and (2.9), with due regard to the calculations carried out at the end of Sect. 2, we obtain

$$\nu(0) = -x_{20} + \sqrt{-x_{10}\cdot\Psi_-(-x_{20}, x_{10})}$$

The line

$$\nu(0) = 0 \tag{4.2}$$

separates the lines (4.1). Under the action of the impulse $\nu(0)\,\delta$ the phase point of system (1.8) is thrown into the range of lines (4.1), if $\nu(0) > 0$, it falls into a position below line (4.2). In case $\nu(0) < 0$, it does not leave the region that is above line (4.2). If the original position lies on line (4.2), the impulse is absent. In any of these versions the subsequent motion is effected along a parabola up to hitting onto the line (4.1) $(i = 2)$. After this the phase point of system (1.8), turning around the origin, leaves the region being considered. Further, with due regard to what was presented at the end of Sect. 2 and in (1.7), we can write the optimal control as a function of the phase coordinates. The extremals of Problem A are shown on Fig. 3. Here, *1* and *2* are the lines (4.1) $(i = 1, 2)$; *3*, *4* are the lines $s_i(x_1) = x_2 - (-1)^i$ $(i = 1, 2)$; *5* is line (4.2).

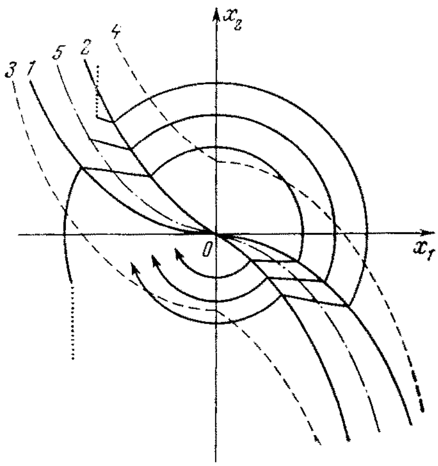

Fig. 3

5. Appendix. Proof of Lemma 4. Allowing for the first of relations (3.1) in the third, we can obtain the expression

$$\psi_{11}^{(i)} = \psi_1^{(i)}(t_1^{(i)}) = (\mu_{11}^{(i)} - 1)\,\mathrm{ctg}\, t_1^{(i)} \quad (i = 1, 2) \tag{5.1}$$

An analysis of (3.2) shows that $t_1^{(1)} < t_1^{(2)}$. This and (5.1) conclude that if $t_1^{(2)} < \pi/2$, then $\psi_{11}^{(1)} > \psi_{11}^{(2)}$ and $\psi_{11}^{(1)} < \psi_{11}^{(2)}$ for $t_1^{(1)} > \pi/2$. The latter signifies that the first line in (2.11) $(t_\beta = t_1^{(1)},\ \psi_1(t_\beta) = \psi_{11}^{(1)}, \psi_2(t_\beta) = \psi_{21}^{(1)},\ \nu(t_\beta) = 1)$ is always to the right of the second. Let us establish that in the half-space $\mu_2 < 0$ the $\psi_{10}^{(i)}$-trajectories are in the relation prescribed by the lemma. From the three cases possible let us consider, for example, that when the first-switching lines are located to the left of the staight line $\mu_1 = 1$. In this case, according to what was presented above, the $\psi_{10}^{(i)}$-trajectories are circles in the region to the right of the first-switching line. The difference of their radii is

$$\Delta r_1 = r_1^{(1)} - r_1^{(2)}, \qquad r_1^{(i)} = \sqrt{[\mu_{21}^{(i)}]^2 - \psi_{11}^{(i)}\mu_{21}^{(i)}}$$

In the region that is to the left of the second line of second switching, the $\psi_{10}^{(i)}$-trajectories are once again circles. the difference of whose radii is $\Delta r_2 = r_2^{(1)} - r_2^{(2)} > 0$. It suffices to establish that $\Delta r_2 > \Delta r_1$. This will be so if $(d/d\alpha)\ \Delta r(\alpha) < 0$, where $\Delta r(\alpha)$ refers to the lines $\mu_1 + \sqrt{-\psi_1^{(i)}\mu_2} = \alpha$, $|\alpha| \leqslant 1$. Computing $(d/d\alpha)\,\mu_2^{(i)} = \mu_2^{(i)\cdot}(\alpha^\cdot)^{-1}$, by virtue of (2.10) $(\psi_2 > 0)$, we obtain

$$\frac{d}{d\alpha}\Delta r(\alpha) = \frac{d}{d\alpha} r^{(1)} - \frac{d}{d\alpha} r^{(2)}, \qquad \frac{d}{d\alpha} r^{(i)} = \sqrt{\psi_1^{(i)}[\psi_1^{(i)} - \mu_2^{(i)}]^{-1}}$$

The components of the gradient of the function under the radical with respect to $\psi_1^{(i)}$, $\mu_2^{(i)}$ are nonnegative. This yields the fact required.

According to the Corollary to Lemma 1, the third-switching lines are located in the half-space $\mu_2 > 0$. It remains to establish that the second of them is to the right of the first. Then, in the region $\mu_2 > 0$ the $\psi_{10}^{(i)}$-trajectories are to be found in the same position as in the half-space $\mu_2 < 0$. Of the three cases possible we again examine only one: the second-switching lines are to the left of the straight line $\mu_1 = -1$, while the third-switching lines are to the right of this same straight line. It is sufficient to establish the diminution of the third function in (2.17) along (2.14) ($\nu = -1$) and the nonpositiveness of $\partial\psi_{13} / \partial l_2$. We have

$$\left.\frac{\partial\psi_{13}}{\partial\mu_{12}}\right|_{(2.14)} = (\tau_3 - l_2)(2 + l_2^2)\, l_2^{-1} \sin \tau_3 < 0$$

since $\tau_3 < l_2$ follows from the inequalities $\psi_{13} < 0$, $\mu_{13} + 1 > 0$. Further

$$\partial\psi_{13} / dl_2 = \mu_{22}\,[(\tau_3' - 1)\ (\mu_{12} + 1)\ \mu_{22}^{-1} + (\tau_3 - l_2)\ (\cos \tau_3 + \mu_{23}\mu_{22}^{-1}\tau_3'] \qquad (5.2)$$

From (2.21)

$$\tau_3' = \tau_3 \sin^2 \tau_3\ (2l_2 - \tau_3)\ [(1 + l_2^2) \sin^2 \tau_3 + \tau_3^2]^{-1}$$

Obviously, $0 < \tau_3' < 1$. By Lemma 1, $\cos \tau_3 < 0$. Consequently, (5.2) is nonpositive.

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CERTAIN PROPERTIES OF PRECESSIONAL MOTIONS RELATIVE TO THE VERTICAL OF A HEAVY SOLID BODY WITH ONE FIXED POINT

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We prove new properties of the precessional motions relative to the vertical of a heavy solid body having a fixed point. In particular, we have shown that semiregular precessions are possible only in the Hesse solution, while in the case when the precession rate and the self-rotation velocity are not constant, the constant of the integral of the angular momentum equals zero.

1. Statement of the problem. Definition [1-3]. The precessional motions of a solid body with one fixed point are the motions under which the angle between two straight lines, one of which is fixed in the body , while the other is fixed in a nonmoving space, remains constant.

Let $\mathbf{k}$ and $\mathbf{v}^*$ be unit vectors fixed, respectively, in the body and in space, and let ϑ be the angle between them. Then, the body's motion is a precession if $\vartheta = \text{const}$. By introducing into consideration the Euler angles ϑ, φ, ψ, we obtain the expression